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LETTER TO THE EDITOR

Radial moments of folding integrals for some non-spherical distributions II

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Abstract. Radial moments of folding integrals for some non-spherical distributions that appear in problems relating to inelastic scattering off nuclei are evaluated in simple closed form in terms of the radial moments of the folded functions.

Satchler (1972) tried to compute the radial moments of some folding integrals which are useful in problems occurring in inelastic scattering off nuclei. He presented special cases but did not examine the general problem. Recently (Rashid 1976) we have presented the computation of the radial moments of the folding integrals in the case where one of the folded functions is a spherically symmetric distribution. Indeed for

$$h(\boldsymbol{r}) = \int f(\boldsymbol{r}_1) g(|\boldsymbol{r}-\boldsymbol{r}_1|)\, \mathrm{d}\boldsymbol{r}_1, \tag{1}$$

defining a multipole expansion by

$$f(\boldsymbol{r}) = \sum_{lm} f_{lm}(r)\, Y_l^m(\Omega) \tag{2}$$

and the radial moments by

$$J_n(f) = 4\pi \int f(r) r^{n+2}\, \mathrm{d}r \tag{3}$$

we obtained (Rashid 1976, equations (8) and (15)):

$$\begin{aligned} J_{l+2k}(h_{lm}) &= 4\pi \int f_{lm}(r_1) g(s)\, Y_l^{m*}(\Omega)\, Y_l^m(\Omega_1) r^{l+2k}\, \mathrm{d}\boldsymbol{r}\, \mathrm{d}\boldsymbol{r}_1 \\ &= \tfrac{1}{2}\sqrt{(\pi)}\, k!\,\Gamma(l+k+\tfrac{3}{2}) \sum_t \frac{J_{l+2k-2t}(f_{lm}) J_{2t}(g)}{t!(k-t)!\,\Gamma(t+\tfrac{3}{2})\Gamma(k+l-t+\tfrac{3}{2})} \end{aligned} \tag{4}$$

where $\boldsymbol{s} = \boldsymbol{r} - \boldsymbol{r}_1$.

In the present paper, we consider the more general case where the spherically symmetric function $g(r)$ is replaced by a multipole $(4\pi)^{1/2} g_{KQ}(r)\, Y_K^Q(\Omega)$ (i.e. by a term in the multipole expansion of $g(\boldsymbol{r})$). The normalization of this term has been chosen to enable us to obtain the special case in equation (4) by replacing (K, Q) by $(0, 0)$. (Note

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that $Y_0^0(\Omega)=(4\pi)^{-1/2}$.) Thus we wish to evaluate

$$J_{L+2k}(h_{LM}^{KQ})=(4\pi)^{3/2}\sum_{lm}\int f_{lm}(r_1)g_{KQ}(s)Y_L^{M*}(\Omega)Y_l^m(\Omega_1)Y_K^Q(\Omega_s)r^{L+2k}\,\mathrm{d}\boldsymbol{r}\,\mathrm{d}\boldsymbol{r}_1. \tag{5}$$

To make comparison with Satchler's work easier, we have used the notations in his paper throughout.

From the structure of the above equation, it is evident that

$$J_{L+2k}(h_{LM}^{KQ})=(4\pi)^{3/2}\sum_l\int f_{l,M-Q}(r_1)g_{KQ}(s)r^{L+2k}$$
$$\times\frac{\langle l,M-Q;K,Q|L,M\rangle}{\langle l,L-K;K,K|L,L\rangle}Y_L^{L*}(\Omega)Y_l^{L-K}(\Omega_1)Y_K^K(\Omega_s)\,\mathrm{d}\boldsymbol{r}\,\mathrm{d}\boldsymbol{r}_1 \tag{6}$$

where

$$|L-K|\leqslant l\leqslant L+K \tag{7}$$

and the Clebsch–Gordan coefficient $\langle l,L-K;K,K|L,L\rangle$ is an extended one which is explicitly given by (Edmonds 1960, equation (3.6.11)):

$$\langle l,L-K;K,K|L,L\rangle=(-1)^{l-L+K}\left(\frac{(2L+1)!(2K)!}{(L-l+K)!(L+l+K+1)!}\right)^{1/2}. \tag{8}$$

To compute the integral in equation (6) above, we use $\boldsymbol{r}_1$ and $\boldsymbol{s}=\boldsymbol{r}-\boldsymbol{r}_1$ as the variables of integration. Thus as in Rashid (1976), we shall require

(i) $$r^2=r_1^2+s^2+2r_1s(\cos\theta_1\cos\theta_s+\sin\theta_1\sin\theta_s\cos(\phi_1-\phi_s)) \tag{9}$$

(ii) $$r^LY_L^{L*}(\Omega)=\frac{(-1)^L}{2^L}\left(\frac{(2L+1)!}{4\pi}\right)^{1/2}\sum_p\frac{r_1^{L-p}s^p}{p!(L-p)!}(\sin\theta_1)^{L-p}(\sin\theta_s)^p\,\mathrm{e}^{\mathrm{i}(p-L)\phi_1-\mathrm{i}p\phi_s}. \tag{10}$$

We also need (Edmonds 1960, equation (3.6.11)):

$$Y_K^K(\Omega_s)=\frac{(-1)^K}{2^KK!}\left(\frac{(2K+1)!}{4\pi}\right)^{1/2}(\sin\theta_s)^K\,\mathrm{e}^{\mathrm{i}K\phi_s} \tag{11}$$

and

$$Y_l^{L-K}(\Omega_1)=\frac{(-1)^{L-K}}{2^ll!}\left(\frac{(2l+1)(-L+l+K)!}{4\pi(L+l-K)!}\right)^{1/2}(\sin\theta_1)^{L-K}$$
$$\times\frac{\mathrm{d}^{L+l-K}}{\mathrm{d}^{L+l-K}(\cos\theta)}(\cos^2\theta-1)^l\,\mathrm{e}^{\mathrm{i}(L-K)\phi_1}$$
$$=\frac{(-1)^{L-K}}{2^l}\left(\frac{(2l+1)(-L+l+K)!}{4\pi(L+l-K)!}\right)^{1/2}(\sin\theta_1)^{L-K}\,\mathrm{e}^{\mathrm{i}(L-K)\phi_1}$$
$$\times\sum_m(-1)^m\frac{(2l-2m)!}{m!(l-m)!\,(2l-2m-L-l+K)!}(\cos\theta)^{-L+l+K-2m}$$
$$=(-1)^{L-K}2^{L-K}\left(\frac{(2l+1)(-L+l+K)!}{4\pi(L+l-K)!}\right)^{1/2}(\sin\theta_1)^{L-K}\,\mathrm{e}^{\mathrm{i}(L-K)\phi_1}$$
$$\times\sum_m(-1)^m\frac{\Gamma(l-m+\frac{1}{2})}{m!\Gamma[\frac{1}{2}(-L+l+K)-m+\frac{1}{2}]\Gamma[\frac{1}{2}(-L+l+K)-m+1]}$$
$$\times(\cos\theta)^{-L+l+K-2m} \tag{12}$$

where (r_1, θ_1, ϕ_1) and (s, θ_s, ϕ_s) are the spherical polar coordinates of $\boldsymbol{r}_1$ and $\boldsymbol{s}$ respectively and we have used the duplication formula

$$\Gamma(2z)=2^{2z-1}\pi^{-1/2}\Gamma(z)\Gamma(z+\tfrac{1}{2}) \tag{13}$$

for the gamma function in the last step in equation (12).

Substituting from equations (8)–(12) in equation (6), we find

$$\begin{aligned}J_{L+2k}(h_{LM}^{KQ}) &= \frac{k!}{2^{2K}K!}\sum_{lmpqtu}(-1)^{-L+l+K}\\&\quad\times\left(\frac{(2l+1)(2K+1)(-L+l+K)!(L-l+K)!(L+l+K+1)!}{(L+l-k)!}\right)^{1/2}\\&\quad\times\frac{2^t\langle l, M-Q; K, Q|L, M\rangle\Gamma(l-m+\frac{1}{2})}{u!p!m!q!(L-p)!(t-u)!(k-q-t)!\Gamma[\frac{1}{2}(-L+l+K)-m+\frac{1}{2}]\Gamma[\frac{1}{2}(-L+l+K)-m+1]}\\&\quad\times\int \mathrm{e}^{\mathrm{i}(p-K)(\phi_1-\phi_s)}(\cos(\phi_1-\phi_s))^u(\sin\theta_1)^{2L-K-p+u+1}(\sin\theta_s)^{K+p+u+1}\\&\quad\times(\cos\theta_1)^{-L+l+K-2m+t-u}(\cos\theta_s)^{t-u}f_{l,M-Q}(r_1)g_{KQ}(s)\\&\quad\times(r_1)^{L+2k-p-2q-t+2}(s)^{p+2q+t+2}\,\mathrm{d}r_1\,\mathrm{d}\theta_1\,\mathrm{d}\phi_1\,\mathrm{d}s\,\mathrm{d}\theta_s\,\mathrm{d}\phi_s.\end{aligned} \tag{14}$$

Using the definition given by equation (3), we can immediately write the radial integrations in terms of the moments of $f_{l,M-Q}$ and g_{KQ} as

$$(4\pi)^{-2}J_{L+2k-p-2q-t}(f_{l,M-Q})J_{p+2q+t}(g_{KQ}).$$

Also the angular integrations give

$$\begin{aligned}&\frac{4\pi^2u!}{2^u\Gamma[\frac{1}{2}(-K+p+u)]\Gamma[\frac{1}{2}(K-p+u)+1]}\\&\quad\times\frac{\Gamma[L-\frac{1}{2}(K+p-u)+1]\Gamma[\frac{1}{2}(-L+l+K+t-u)-m+\frac{1}{2}]}{\Gamma[\frac{1}{2}(L+l-p+t)-m+\frac{3}{2}]}\\&\quad\times\frac{\Gamma[\frac{1}{2}(K+p+u)+1]\Gamma[\frac{1}{2}(t-u)+\frac{1}{2}]}{\Gamma[\frac{1}{2}(K+p+t)+\frac{3}{2}]}\end{aligned}$$

where $-L+l+K-2m+t-u$ and $t-u$ are both non-negative integers on account of the presence of factorials in equation (14). The non-negativity of the arguments of the factorials present determines the ranges of summation of the summation indices (this is the reason why these ranges are omitted everywhere in the paper). This shows that all angular integrations are convergent. For non-zero values of the angular integrations we require that $-K+p+u$, $-L+l+K-2m+t-u$ and $t-u$ must also be even (and non-negative). In other words, all the arguments of the gamma functions are integers except for the presence of $\frac{1}{2}$ and $\frac{3}{2}$. Writing the gamma functions as factorials where the arguments do not contain $\frac{1}{2}$ or $\frac{3}{2}$, we arrive at:

$$\begin{aligned}&J_{L+2k}(h_{LM}^{KQ})\\&\quad=\frac{\sqrt{(\pi)}}{2^{2K+2}}\frac{k!}{K!}\sum_{lmpqtu}\left(\frac{(2l+1)(2K+1)(-L+l+K)!(L-l+K)!(L+l+K+1)!}{(L+l-K)!}\right)^{1/2}\\&\quad\times\langle l, M-Q; K, Q|L, L\rangle(-1)^m\end{aligned}$$

$$\times\frac{\Gamma(l-m+\frac{1}{2})[L-\frac{1}{2}(K+p-u)]!\Gamma[\frac{1}{2}(-L+l+K+t-u)-m+\frac{1}{2}]}{p!(L-p)!m!\Gamma[\frac{1}{2}(-L+l+K)-m+\frac{1}{2}][\frac{1}{2}(-L+l+K)-m]!}$$

$$\times\frac{[\frac{1}{2}(K+p+u)]!J_{L+2k-p-2q-t}(f_{l,M-Q})J_{p+2q+t}(g_{KQ})}{q![\frac{1}{2}(t-u)]!(k-q-t)![\frac{1}{2}(-K+p+u)]![\frac{1}{2}(K-p+u)]!\times\Gamma[\frac{1}{2}(L+l-p+t)-m+\frac{3}{2}]\Gamma[\frac{1}{2}(K+p+t)+\frac{3}{2}]}. \quad (15)$$

Now $p+2q+t-K=(t-u)+(-K+p+u)+2q$ is an even non-negative integer. Thus we can replace the summation index t by $K-p-2q+2t$ where the new t is a non-negative integer. We also replace $(K-p+u)$ by $2u$ and $k-q-t$ by q. These three transformations effectively mean the replacement of q, t and u by $-k+K-p+q+2t$, $2k-K+p-2q-2t$ and $-K+p+2u$ respectively. This results in

$$J_{L+2k}(h^{KQ}_{LM})$$
$$=\frac{\sqrt{(\pi)}k!}{2^{2K+2}K!}\sum_{l,t}\left(\frac{(2l+1)(2K+1)(-L+l+K)!(L-l+K)!(L+l+K+1)!}{(L+l-K)!}\right)^{1/2}$$
$$\times\langle l,M-Q;K,Q|L,L\rangle J_{L+2k-K-2t}(f_{l,M-Q})J_{K+2t}(g_{KQ})$$
$$\times\sum_{mq}\frac{(-1)^m\Gamma(l-m+\frac{1}{2})}{m!\Gamma[\frac{1}{2}(-L+l+K)-m+\frac{1}{2}][\frac{1}{2}(-L+l+K)-m]!q!\Gamma[k+\frac{1}{2}(L+l-K)-m-q-t+\frac{3}{2}]}$$
$$\times\sum_{pu}\frac{(L-K+u)!\Gamma[k+\frac{1}{2}(-L+l+K)-m-q-t-u+\frac{1}{2}](p+u)!}{p!(L-p)!(-k+K-p+q-2t)!(k-q-t-u)!(-K+p+u)!u!\Gamma(k+p-q-t+\frac{3}{2})}. \quad (16)$$

In the following appendix, we show how the part in the above equation involving *mqpu* summations can be simplified to give the expression in equation (A.6). Thus the above equation takes the form

$$J_{L+2k}(h^{KQ}_{LM})=\frac{\sqrt{(\pi)}k!\Gamma(k+L+\frac{3}{2})}{2^{2K+2}}$$
$$\times\sum_{l}(-1)^{\frac{1}{2}(-L+l+K)}$$
$$\times\left(\frac{(2l+1)(2K+1)(-L+l+K)!(L-l+K)!(L+l+K+1)!}{(L+l-K)!}\right)^{1/2}$$
$$\times\frac{\Gamma[\frac{1}{2}(L+l-K)+\frac{1}{2}]}{[\frac{1}{2}(-L+l+K)]![\frac{1}{2}(L-l+K)]!\Gamma[\frac{1}{2}(L+l+K)+\frac{3}{2}]}$$
$$\times\sum_{t}\frac{1}{t!\Gamma(t+K+\frac{3}{2})\Gamma[k+\frac{1}{2}(L+l-K)-t+\frac{3}{2}][k+\frac{1}{2}(L-l-K)-t]!}. \quad (17)$$

Using the definition of the special Clebsch–Gordan coefficient $\langle l,0;K,0|L,L\rangle$ and the duplication formula (equation (13)), we arrive at the final expression for $J_{L+2k}(h^{KQ}_{LM})$ as

$$J_{L+2k}(h^{KQ}_{LM})=\tfrac{1}{2}\sqrt{(\pi)}k!\Gamma(k+L+\tfrac{3}{2})$$
$$\times\sum_{l}\left(\frac{(2l+1)(2K+1)}{(2L+1)}\right)^{1/2}\langle l,M-Q;K,Q|L,L\rangle\langle l,0;K,0|L,0\rangle$$
$$\times\sum_{t}\frac{J_{L+2k-K-2t}(f_{l,M-Q})g_{K+2t}(g_{KQ})}{t!\Gamma(K+t+\frac{3}{2})[k+\frac{1}{2}(L-l-K)-t]!\Gamma[k+\frac{1}{2}(L+l-K)-t+\frac{3}{2}]}. \quad (18)$$

This is our final result which relates the radial moment of the folding integral with those of the folded functions.

In conclusion, we present a few remarks.

(i) The special case of Rashid (1976) given in equation (4) is immediately obtained by replacing (K, Q) by $(0, 0)$ in which case the index l takes the only value L.

(ii) Putting $k=0$, we note that $\frac{1}{2}(L-l-K)-t\geqslant 0$ requires $l=L-K$, $t=0$. One then obtains the special case given by Satchler (1972, equation (25)).

(iii) The presence of the factorial $[k+\frac{1}{2}(L-l-K)-r]!$ requires $L+2k-K-2r\geqslant l$. This condition was not clear at any stage before but it shows that $J_{L+2k}(h^{KQ}_{LM})$ is related to the moments $J_n(f_{l,M-Q})$ of $f_{l,M-Q}$ with $n\geqslant l$ necessarily.

Appendix

We wish to compute the *mpqu* summation appearing in equation (16) (denoted by S below). Let us first examine the *pu* summation only. Since (Edmonds 1960, equation (A.1.1))

$$\frac{(p+u)!}{K!p!(-K+p+u)!u!}=\sum_r \frac{1}{r!(u-K+r)!(p-r)!(K-r)!}, \tag{A.1}$$

the part containing the p, u summations takes the form

$$K!\sum_r \frac{1}{r!(K-r)!}$$
$$\times\sum_p \frac{1}{(L-p)!(p-r)!\Gamma(k+p-q-t+\frac{3}{2})(-k+K-p+q+2r)!}$$
$$\times\sum_u \frac{(L-K+u)!\Gamma[k+\frac{1}{2}(-L+l+K)-m-q-t-u+\frac{1}{2}]}{(-K+r+u)!(k-q-t-u)!} \tag{A.2}$$

which becomes, on performing the p, u summations using equations (A1.1) and (A1.3) of Edmonds (1960):

$$K!\frac{\Gamma[k+\frac{1}{2}(L+l-K)-m-q-t+\frac{3}{2}]\Gamma[\frac{1}{2}(-L+l+K)-m+\frac{1}{2}]}{\Gamma(K+t+\frac{3}{2})\Gamma(k+L-q-t+\frac{3}{2})}$$
$$\times\sum \frac{\Gamma(L+K+t-r+\frac{3}{2})}{r!(K-r)!(-k+K+q+2t-r)!(k-K-q-t+r)!\Gamma[\frac{1}{2}(L+l+K)-m-r+\frac{3}{2}]}. \tag{A.3}$$

On substituting from (A.3) in the *pu* summation in equation (16), the quantity S becomes

$$S=\frac{K!}{\Gamma(K+t+\frac{3}{2})}\sum_r \frac{\Gamma(L+K+t-r+\frac{3}{2})}{r!(K-r)!}$$
$$\times\sum_m (-1)^m \frac{\Gamma(l-m+\frac{1}{2})}{m![\frac{1}{2}(-L+l+K)-m]!\Gamma[\frac{1}{2}(L+l+K)-m-r+\frac{3}{2}]}$$
$$\times\sum_q \frac{1}{q!\Gamma(k+L-q-t+\frac{3}{2})(-k+K+q+2t-r)!(k-K-q-t+r)!} \tag{A.4}$$

where now the m and q summations can be performed using equations (A1.2) and (A1.1) of Edmonds (1960) respectively. Thus

$$S=(-1)^{\frac{1}{2}(-L+l+K)}\frac{K!\Gamma(k+L+\frac{3}{2})\Gamma[\frac{1}{2}(L+l-K)+\frac{1}{2}]}{[\frac{1}{2}(-L+l+K)]!t!\Gamma(K+t+\frac{3}{2})\Gamma(k+L-t+\frac{3}{2})}$$
$$\times\sum_r\frac{1}{r!(k-K+r-t)!\Gamma[\frac{1}{2}(L+l+K)-r+\frac{3}{2}][\frac{1}{2}(L-l+K)-r]!}\qquad\text{(A.5)}$$

where the r summation can also be performed on making use of equation (A1.1) of Edmonds (1960) which leads to

$$S=\frac{(-1)^{\frac{1}{2}(-L+l+K)}K!\Gamma(k+L+\frac{3}{2})\Gamma[\frac{1}{2}(L+l-K)+\frac{1}{2}]}{[\frac{1}{2}(-L+l+K)![\frac{1}{2}(L-l+K)]!\Gamma[\frac{1}{2}(L+l+K)+\frac{3}{2}]t!\Gamma(K+t+\frac{3}{2})\times[k+\frac{1}{2}(L-l-K)-t]!\Gamma[k+\frac{1}{2}(L+l-K)-t+\frac{3}{2}]}\qquad\text{(A.6)}$$

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